Analytic Treatment for (2+1)-Dimensional Kortweg-de Vries-Like and Kadomtsev-Petviashvili-Like Equations

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In this work we present a reliable treatment for two (2+1)-dimensional Korteweg-de Vries-like and Kadomtsev-Petviashvili-like equations. The Hirota bilinear method will be used to show that these two equations are not completely integrable equations. Unlike the completely integrable Korteweg-de Vries and Kadomtsev-Petviashvili equations, where multiple soliton solutions exist, only one-soliton and two-soliton solutions can be derived for each of the Korteweg-de Vries-like and Kadomtsev-Petviashvili-like equations.

Key words: Hirota Bilinear Method; Korteweg-de Vries Equation; Kadomtsev-Petviashvili-Like Equation.

1. Introduction

It is well-known that the Korteweg-de Vries (KdV) equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{1}$$

can be expressed in terms of the bilinear operators [1 – 6]

$$\mathrm{D}_x(\mathrm{D}_t + \mathrm{D}_x^3) f \cdot f = 0. \tag{2}$$

The KdV equation (1) models a variety of nonlinear phenomena, including ion acoustic waves in plasmas and shallow water waves. The nonlinear term uu_x describes the steepening of the wave and the linear term u_{xxx} accounts for the spreading or dispersion of the wave.

On the other hand, the Kadomtsev-Petviashvili (KP) equation [7] extends the KdV equation, and is given by

$$(u_t + 6uu_x + u_{xxx})_x + u_{yy} = 0, \tag{3}$$

which can be expressed in terms of the bilinear operators as

$$\left[\mathrm{D}_x(\mathrm{D}_t + \mathrm{D}_x^3) + \mathrm{D}_y^2\right] f \cdot f = 0. \tag{4}$$

The KP equation (3) is used to model shallow-water waves with weakly nonlinear restoring forces. It is a natural generalization of the KdV equation from (1+1) to (2+1) dimensions.

The Hirota bilinear operators [1] are defined by the following rule:

$$\mathrm{D}_t^n \mathrm{D}_x^m a \cdot b = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \cdot a(x,t) b(x',t')|_{x'=x, t'=t}. \tag{5}$$

The solution of (1) is of the form

$$u(x,t) = 2(\ln f(x,t))_{xx}, \tag{6}$$

where the auxiliary function $f(x,t)$ has a perturbation expansion of the form

$$f(x,t) = 1 + \sum_{n=1}^{\infty} \varepsilon^n f_n(x,t). \tag{7}$$

The solution of (3) is of the form

$$u(x,y,t) = 2(\ln f(x,y,t))_{xx}, \tag{8}$$

where the auxiliary function $f(x,y,t)$ has a perturbation expansion of the form

$$f(x,y,t) = 1 + \sum_{n=1}^{\infty} \varepsilon^n f_n(x,y,t). \tag{9}$$

It is well known that the KdV equation and the KP equation are completely integrable and multiple soliton solutions exist for each equation.

The nonlinear evolution equations attracted a huge size of research works to establish a variety of solutions of distinct physical structures [8 – 23]. Several methods have been developed aiming to achieve an useful progress by developing more solutions and to facilitate the calculations. Such methods are the Darboux transformation, the Lie group method, the variable separation method, the Painlevé analysis method, the exp-function method, the sine-cosine method, the Jacobi elliptic function method, and various tanh function methods. The tanh-coth method provides single solitons and plane periodic solutions as well. Computer symbolic systems such as Maple and Mathematica allow us to perform complicated and tedious calculations.

In this work we study a (2+1)-dimensional KdV-like equation in the form [24 – 25]

$$u_t + \alpha u_x u_y + u_{xxy} = 0. \tag{10}$$

In addition, we will study another (2+1)-dimensional KP-like equation in the form [24 – 26]

$$(u_t + \alpha u_x u_y + u_{xxy})_x + u_{yy} = 0. \tag{11}$$

The main goal of this work will be on examining the complete integrability of these two equations. Our work depends mainly on the Hirota bilinear method [1 – 4].

2. The Hirota Bilinear Method

The Hirota direct method is well known, and it gives soliton solutions by polynomials of exponentials. We only summarize the necessary steps, where details can be found in [1 – 16] among many others.

We first substitute

$$u(x,y,t) = \mathrm{e}^{kx+ky-ct} \tag{12}$$

into the linear terms of the equation under discussion to determine the dispersion relation between m,n, and r. We then substitute the single soliton solution

$$u(x,y,t) = R(\ln f(x,y,t))_{xx} \tag{13}$$

into the equation under discussion, where the auxiliary function $f(x,y,t)$ is given by

$$f(x,y,t) = 1 + C_1 f_1(x,y,t) = 1 + C_1 \mathrm{e}^{\theta_i} \tag{14}$$

with

$$\theta_i = k_i x + r_i y - c_i t, i = 1,2,\cdots,N. \tag{15}$$

Solving the resulting equation determines the numerical value for R. Notice that the N-soliton solutions can be obtained by using the following forms for $f(x,y,t)$ into (13):

(i) For dispersion relation, we use

$$u(x,y,t) = \mathrm{e}^{\theta_i}, \theta_i = k_i x + r_i y - c_i t. \tag{16}$$

(ii) For single soliton, we use

$$f(x,y,t) = 1 + C_1 \mathrm{e}^{\theta_1}. \tag{17}$$

(iii) For two-soliton solutions, we use

$$f(x,y,t) = 1 + C_1 \mathrm{e}^{\theta_1} + C_2 \mathrm{e}^{\theta_2} + C_1 C_2 a_{12} \mathrm{e}^{\theta_1+\theta_2}. \tag{18}$$

(iv) For three-soliton solutions, we use

$$\begin{aligned} f(x,y,t) &= 1 + C_1 \mathrm{e}^{\theta_1} + C_2 \mathrm{e}^{\theta_2} + C_3 \mathrm{e}^{\theta_3} \\ &+ C_1 C_2 a_{12} \mathrm{e}^{\theta_1+\theta_2} + C_2 C_3 a_{23} \mathrm{e}^{\theta_2+\theta_3} \\ &+ C_1 C_3 a_{13} \mathrm{e}^{\theta_1+\theta_3} + C_1 C_2 C_3 b_{123} \mathrm{e}^{\theta_1+\theta_2+\theta_3}. \end{aligned} \tag{19}$$

Notice that we use (16) to determine the dispersion relation, (18) to determine the phase shift a_{12} to be generalized for the other factors a_{ij}, and finally we use (19) to determine b_{123}, which is given by $b_{123} = a_{12} a_{23} a_{13}$ for completely integrable equations. The determination of three-soliton solutions, if they exist, confirms the fact that N-soliton solutions exist for any order.

3. The (2+1)-Dimensional KdV-Like Equation

We begin our analysis by studying the (2+1)-dimensional KdV-like equation

$$u_t + \alpha u_x u_y + u_{xxy} = 0. \tag{20}$$

We first consider $C_1 = C_2 = C_3 = +1$. Substituting

$$u(x,y,t) = \mathrm{e}^{\theta_i}, \theta_i = k_i x + r_i y - c_i t, \tag{21}$$

into the linear terms of (20) gives the dispersion relation

$$c_i = r_i k_i^2,\ i = 1,2,\cdots N \tag{22}$$

and hence θ_i becomes

$$\theta_i = k_i x + r_i y - r_i k_i^2 t. \tag{23}$$

To determine R, we substitute

$$u(x,y,t) = R(\ln f(x,y,t))_x, \tag{24}$$

where $f(x,y,t) = 1 + e^{k_1x+r_1y-r_1k_1^2t}$ into (20) and find that

$$R = \frac{6}{\alpha}, \quad \alpha \neq 0. \tag{25}$$

This means that the single-kink solution is given by

$$u(x,y,t) = \frac{6k_1 e^{k_1x+k_1y-r_1k_1^2t}}{\alpha(1+e^{k_1x+r_1y-r_1k_1^2t})}. \tag{26}$$

For two-kink solutions, we substitute

$$u(x,y,t) = \frac{6}{\alpha}(\ln f(x,y,t))_x, \tag{27}$$

where

$$f = 1 + e^{\theta_1} + e^{\theta_2} + a_{12}e^{\theta_1+\theta_2}, \tag{28}$$

into (20) to find that the phase shift is given by

$$a_{12} = \frac{(k_1-k_2)(k_1^2r_2+2k_1k_2(r_1-r_2)-k_2^2r_1)}{(k_1+k_2)(k_1^2r_2+2k_1k_2(r_1+r_2)+k_2^2r_1)}, \tag{29}$$

and hence we can generalize

$$a_{ij} = \frac{(k_i-k_j)(k_i^2r_j+2k_ik_j(r_i-r_j)-k_j^2r_i)}{(k_i+k_j)(k_i^2r_j+2k_ik_j(r_i+r_j)+k_j^2r_i)}, \quad 1 \leq i < j \leq N. \tag{30}$$

This in turn gives

$$\begin{aligned} f(x,y,t) &= 1 + e^{k_1x+r_1y-r_1k_1^2t} + e^{k_2x+r_2y-r_2k_2^2t} \\ &+ \frac{(k_1-k_2)(k_1^2r_2+2k_1k_2(r_1-r_2)-k_2^2r_1)}{(k_1+k_2)(k_1^2r_2+2k_1k_2(r_1+r_2)+k_2^2r_1)} \\ &\cdot e^{(k_1+k_2)x+(r_1+r_2)y-(r_1k_1^2+r_2k_2^2)t}. \end{aligned} \tag{31}$$

To determine the two-kink solutions explicitly, we substitute the last result for $f(x,y,t)$ into (27).

Following the discussion presented before, we try to find the three-kink solutions, therefore we set

$$\begin{aligned} f(x,y,t) &= 1 + e^{(\theta_1)} + e^{(\theta_2)} + e^{(\theta_3)} + a_{12}e^{(\theta_1+\theta_2)} \\ &+ a_{23}e^{(\theta_2+\theta_3)} + a_{13}e^{(\theta_1+\theta_3)} + b_{123}e^{(\theta_1+\theta_2+\theta_3)} \end{aligned} \tag{32}$$

into (27) and substitute it in (20) to find that

$$b_{123} \neq a_{12}a_{13}a_{23}. \tag{33}$$

In view of the last result we cannot find three-kink solutions. We therefore conclude that the (2+1)-dimensional KdV-like equation (20) is not completely integrable and does not possess multiple-soliton solutions of any order $N \geq 3$. This shows that the conclusion made in [25] that equation (20) is integrable is not correct.

We next consider the case where $C_1 = C_2 = C_3 = -1$. Proceeding as before, using the dispersion relation

$$c_i = r_ik_i^2, i = 1,2,\cdots N \tag{34}$$

and substituting

$$u(x,y,t) = R(\ln f(x,y,t))_{xy}, \tag{35}$$

where $f(x,y,t) = 1 - e^{k_1x+r_1y-r_1k_1^2t}$ into (20), we find that

$$R = \frac{6}{\alpha}, \quad \alpha \neq 0. \tag{36}$$

This means that the single singular-kink solution is given by

$$u(x,y,t) = -\frac{6k_1 e^{k_1x+r_1y-r_1k_1^2t}}{\alpha(1-e^{k_1x+r_1y-r_1k_1^2t})}. \tag{37}$$

For two singular-kink solutions, we substitute

$$f = 1 - e^{\theta_1} - e^{\theta_2} + a_{12}e^{\theta_1+\theta_2} \tag{38}$$

into (20) to obtain that

$$a_{12} = \frac{(k_1-k_2)(k_1^2r_2+2k_1k_2(r_1-r_2)-k_2^2r_1)}{(k_1+k_2)(k_1^2r_2+2k_1k_2(r_1+r_2)+k_2^2r_1)}, \tag{39}$$

and hence we can generalize

$$a_{ij} = \frac{(k_i-k_j)(k_i^2r_j+2k_ik_j(r_i-r_j)-k_j^2r_i)}{(k_i+k_j)(k_i^2r_j+2k_ik_j(r_i+r_j)+k_j^2r_i)}, \quad 1 \leq i < j \leq N. \tag{40}$$

This in turn gives

$$\begin{aligned} f(x,y,t) &= 1 - e^{k_1x+r_1y-r_1k_1^2t} - e^{k_2x+r_2y-r_2k_2^2t} \\ &+ : \frac{(k_1-k_2)(k_1^2r_2+2k_1k_2(r_1-r_2)-k_2^2r_1)}{(k_1+k_2)(k_1^2r_2+2k_1k_2(r_1+r_2)+k_2^2r_1)} \\ &\cdot e^{(k_1+k_2)x+(k_1+k_2)y-(r_1k_1^2+r_2k_2^2)t}. \end{aligned} \tag{41}$$

To determine the two singular-kink solutions explicitly, we substitute the last result for $f(x,y,t)$ into (27).

Following the discussion presented before, we can determine the three-soliton solutions. Therefore we set

$$f(x,y,t) = 1 - e^{(\theta_1)} - e^{(\theta_2) - e(\theta_3)} + a_{12}e^{(\theta_1+\theta_2)} + a_{23}e^{(\theta_2+\theta_3)} + a_{13}e^{(\theta_1+\theta_3)} + b_{123}e^{(\theta_1+\theta_2+\theta_3)} \tag{42}$$

into (27) and substitute it in (20) to find that

$$b_{123} \neq -a_{12}a_{13}a_{23}. \tag{43}$$

This clearly means that we cannot derive three singular-kink solutions.

4. The (2+1)-Dimensional KP-Like Equation

We next consider the (2+1)-dimensional KP-like equation

$$(u_t + \alpha u_x u_y + u_{xxy})_x + u_{yy} = 0. \tag{44}$$

We first start with the case where $C_1 = C_2 = C_3 = +1$. Substituting

$$u(x,y,t) = e^{\theta_i}, \theta_i = k_i x + r_i y - c_i t, \tag{45}$$

into the linear terms of (44) gives the dispersion relation

$$c_i = \frac{r_i k_i^3 + r_i^2}{k_i}, \quad i = 1, 2, \cdots N, \tag{46}$$

and hence θ_i becomes

$$\theta_i = k_i x + r_i y - \frac{r_i k_i^3 + r_i^2}{k_i} t. \tag{47}$$

To determine R, we substitute

$$u(x,y,t) = R(\ln f(x,y,t))_x, \tag{48}$$

where $f(x,y,t) = 1 + e^{k_1 x + r_1 y - \frac{r_1 k_1^3 + r_1^2}{k_1} t}$ into (44) and find that

$$R = \frac{6}{\alpha}, \quad \alpha \neq 0. \tag{49}$$

This means that the single-kink solution is given by

$$u(x,y,t) = \frac{6k_1 e^{k_1 x + k_1 y - \frac{r_1 k_1^3 + r_1^2}{k_1} t}}{\alpha\left(1 + e^{k_1 x + r_1 y - \frac{r_1 k_1^3 + r_1^2}{k_1} t}\right)}. \tag{50}$$

For two-kink solutions, we substitute

$$u(x,y,t) = \frac{6}{\alpha}(\ln f(x,y,t))_x, \tag{51}$$

where

$$f = 1 + e^{\theta_1} + e^{\theta_2} + a_{12}e^{\theta_1+\theta_2}, \tag{52}$$

into (44) and find that the phase shift a_{12} is given by

$$a_{12} = \frac{k_1k_2(r_1k_2^3 + r_2k_1^3) - (r_1^2k_2^2 + r_2^2k_1^2) - 3k_1^2k_2^2(r_1k_2 + r_2k_1) + 2k_1^2k_2^2(r_2k_2 + r_1k_1) + 2r_1r_2k_1k_2}{k_1k_2(r_1k_2^3 + r_2k_1^3) - (r_1^2k_2^2 + r_2^2k_1^2) + 3k_1^2k_2^2(r_1k_2 + r_2k_1) + 2k_1^2k_2^2(r_2k_2 + r_1k_1) + 2r_1r_2k_1k_2}, \tag{53}$$

and hence we can generalize

$$a_{ij} = \frac{k_ik_j(r_ik_j^3 + r_jk_i^3) - (r_i^2k_j^2 + r_j^2k_i^2) - 3k_i^2k_j^2(r_ik_j + r_jk_i) + 2k_i^2k_j^2(r_jk_j + r_ik_i) + 2r_ir_jk_ik_j}{k_ik_j(r_ik_j^3 + r_jk_i^3) - (r_i^2k_j^2 + r_j^2k_i^2) + 3k_i^2k_j^2(r_ik_j + r_2k_i) + 2k_i^2k_j^2(r_jk_j + r_ik_i) + 2r_ir_jk_ik_j} \tag{54}$$

for $1 \leq i < j \leq N$.
This in turn gives

$$\begin{aligned} f(x,y,t) = {} & 1 + e^{k_1 x + r_1 y - \frac{r_1 k_1^3 + r_1^2}{k_1} t} + e^{k_2 x + r_2 y - \frac{r_2 k_2^3 + r_2^2}{k_2} t} \\ & + \frac{k_1k_2(r_1k_2^3 + r_2k_1^3) - (r_1^2k_2^2 + r_2^2k_1^2) - 3k_1^2k_2^2(r_1k_2 + r_2k_1) + 2k_1^2k_2^2(r_2k_2 + r_1k_1) + 2r_1r_2k_1k_2}{k_1k_2(r_1k_2^3 + r_2k_1^3) - (r_1^2k_2^2 + r_2^2k_1^2) + 3k_1^2k_2^2(r_1k_2 + r_2k_1) + 2k_1^2k_2^2(r_2k_2 + r_1k_1) + 2r_1r_2k_1k_2} \\ & \cdot e^{(k_1+k_2)x + (r_1+r_2)y - (\frac{r_1k_1^3 + r_1^2}{k_1} + \frac{r_2k_2^3 + r_2^2}{k_2})t}. \end{aligned} \tag{55}$$

To determine the two-kink solutions explicitly, we substitute the last result for $f(x,y,t)$ into (51).

We finally try to find the three-kink solutions. Therefore we set

$$f(x,y,t) = 1 + \mathrm{e}^{(\theta_1)} + \mathrm{e}^{(\theta_2)} + \mathrm{e}^{(\theta_3)} + a_{12}\mathrm{e}^{(\theta_1+\theta_2)} + a_{23}\mathrm{e}^{(\theta_2+\theta_3)} + a_{13}\mathrm{e}^{(\theta_1+\theta_3)} + b_{123}\mathrm{e}^{(\theta_1+\theta_2+\theta_3)} \tag{56}$$

into (51) and substitute it in (44) to find that

$$b_{123} \neq a_{12}a_{13}a_{23}. \tag{57}$$

In view of the last result we cannot find three-kink solutions. We therefore conclude that the (2+1)-dimensional KP-like equation (44) is not completely integrable and does not possess multiple-soliton solutions of any order $N \geq 3$.

To find the singular soliton solutions, we proceed as before, hence, we list the results only. The single singular-kink solution is given by

$$u(x,y,t) = -\frac{6k_1\mathrm{e}^{k_1x+r_1y-\frac{r_1k_1^3+r_1^2}{k_1}t}}{\alpha\left(1-\mathrm{e}^{k_1x+r_1y-\frac{r_1k_1^3+r_1^2}{k_1}t}\right)}. \tag{58}$$

For two singular-kink solutions, we find

$$\begin{aligned} f(x,y,t) = {} & 1 - \mathrm{e}^{k_1x+r_1y-\frac{r_1k_1^3+r_1^2}{k_1}t} - \mathrm{e}^{k_2x+r_2y-\frac{r_2k_2^3+r_2^2}{k_2}t} \\ & + \frac{k_1k_2(r_1k_2^3+r_2k_1^3)-(r_1^2k_2^2+r_2^2k_1^2)-3k_1^2k_2^2(r_1k_2+r_2k_1)+2k_1^2k_2^2(r_2k_2+r_1k_1)+2r_1r_2k_1k_2}{k_1k_2(r_1k_2^3+r_2k_1^3)-(r_1^2k_2^2+r_2^2k_1^2)+3k_1^2k_2^2(r_1k_2+r_2k_1)+2k_1^2k_2^2(r_2k_2+r_1k_1)+2r_1r_2k_1k_2} \\ & \cdot \mathrm{e}^{(k_1+k_2)x+(r_1+r_2)y-\left(\frac{r_1k_1^3+r_1^2}{k_1}+\frac{r_2k_2^3+r_2^2}{k_2}\right)t}. \end{aligned} \tag{59}$$

To determine the two singular-kink solutions explicitly, we substitute the last result for $f(x,y,t)$ into (51). The three singular-kink solutions cannot be obtained.

5. Discussion

The (2+1)-dimensional KdV-like and KP-like equations were examined. The standard KdV and the KP equations give rise to multiple soliton solutions. However, the KdV-like and the KP-like equations give only one-soliton and two-soliton solutions. This shows that the two equations are not completely integrable. The powerful Hirota bilinear method was used to justify this goal.

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